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The bisymmetric solutions of the matrix equation $A_1X_1B_1 + A_2X_2B_2 + \cdots + A_lX_lB_l = C$ and its optimal approximation ☆

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Abstract

A matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be bisymmetric matrix if $a_{ij} = a_{ji} = a_{n+1-j,n+1-i}$ for all $1 \leqslant i, j \leqslant n$. In this paper, an iterative method is constructed to find the bisymmetric solutions of matrix equation $A_1X_1B_1 + A_2X_2B_2 + \cdots + A_lX_lB_l = C$ where $[X_1, X_2, \ldots, X_l]$ is real matrices group. By this iterative method, the solvability of the matrix equation can be judged automatically. When the matrix equation is consistent, for any initial bisymmetric matrix group $[X_1^{(0)}, X_2^{(0)}, \ldots, X_l^{(0)}]$, a bisymmetric solution group can be obtained within finite iteration steps in the absence of roundoff errors, and the least norm bisymmetric solution group can be obtained by choosing a special kind of initial bisymmetric matrix group. In addition, the optimal approximation bisymmetric solution group to a given bisymmetric matrix group $[\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_l]$ in Frobenius norm can be obtained by finding the least norm bisymmetric solution group of new matrix equation $A_1\widetilde{X}_1B_1 + A_2\widetilde{X}_2B_2 + \cdots + A_l\widetilde{X}_lB_l = \widetilde{C}$, where $\widetilde{C} = C - A_1\overline{X}_1B_1 - A_2\overline{X}_2B_2 - \cdots - A_l\overline{X}_lB_l$.

Keywords: Iterative method; Matrix equation; Bisymmetric solution group; Least-norm solution group; Optimal approximation solution

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1. Introduction

A matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be bisymmetric matrix if $a_{ij} = a_{ji} = a_{n+1-j,n+1-i}$ for all $1 \leqslant i, j \leqslant n$. Let $R^{m\times n}$, $SR^{n\times n}$ and $BSR^{n\times n}$ denote the set of $m \times n$ real matrices, $n \times n$ real symmetric matrices and $n \times n$ real bisymmetric matrices respectively. $S_n(S_n = (e_n, e_{n-1}, \dots, e_1))$ denotes the $n \times n$ reverse unit matrix (e_i denotes ith column of $n \times n$ unit matrix). The superscripts T and + represent the transpose and Moore–Penrose generalized inverse of a matrix respectively. In space $R^{m\times n}$, we define inner product as: $\langle A, B\rangle = \mathrm{trace}(B^{\mathrm{T}}A)$ for all $A, B \in R^{m\times n}$. Then the norm of a matrix A generated by this inner product is, obviously, Frobenius norm and denoted by $\|A\|$.

In this paper, we consider the following two problems:

Problem I. Given $A_i \in R^{p\times n_i}$, $B_i \in R^{n_i\times q} (i = 1, 2, \dots, l)$ and $C \in R^{p\times q}$, find matrix group $[X_1, X_2, \dots, X_l]$ with $X_i \in BSR^{n_i\times n_i}$, $i = 1, 2, \dots, l$, such that

$$A_1X_1B_1 + A_2X_2B_2 + \cdots + A_lX_lB_l = C. \tag{1.1}$$

Problem II. When Problem I is consistent, let S_E denote its solution group set, for given matrix group $[\overline{X}_1, \overline{X}_2, \dots, \overline{X}_l]$ with $\overline{X}_i \in BSR^{n_i\times n_i}$ $(i = 1, 2, \dots, l)$, find $[\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_l] \in S_E$ with $\widehat{X}_i \in BSR^{n_i\times n_i}$, such that

$$\begin{aligned}&\|\widehat{X}_1 - \overline{X}_1\|^2 + \|\widehat{X}_2 - \overline{X}_2\|^2 + \cdots + \|\widehat{X}_l - \overline{X}_l\|^2\\ &\quad = \min_{[X_1,X_2,\dots,X_l]\in S_E} [\|X_1 - \overline{X}_1\|^2 + \|X_2 - \overline{X}_2\|^2 + \cdots + \|X_l - \overline{X}_l\|^2].\end{aligned}$$

Various linear matrix equations have been investigated. For example, Dai [6] has studied the linear matrix equation $AXB = C$ with a symmetric condition on the solution, Peng [1,2], Shim [3], Chu [4] have studied the linear matrix equation $AXB + CYD = E$ with unknown matrices X and Y being real or complex. The methods used in these papers included generalized inverse, generalized singular value decomposition (GSVD) and canonical correlation decomposition (CCD) of matrices. In these papers, the necessary and sufficient conditions for the existence of and the expressions for the solution of the equation were established. Peng [5] has researched the equation $A_1X_1B_1 + A_2X_2B_2 + \cdots + A_kX_kB_k = D$ with the bisymmetric conditions on the solutions. Because it is difficult to solve this equation by using the matrix decomposition methods in [1–4,6], vec operator and Kronecker product are employed to solve the matrix equation by Peng [5]. However, the size of the matrix is enlarged greatly and the computation is very expensive in the process of solving solutions, namely, the existence conditions of the solution are examined and the equation is solved, both of which are very complicated. This method in [5] suit only to solve matrix equations of small size.

Problem II occurs frequently in experimental design. Here the matrix group $[\overline{X}_1, \overline{X}_2, \dots, \overline{X}_l]$ may be obtained from experiments, but it may not be the solution group of Problem I. The best estimate $[\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_l]$ is the matrix group that not only is the solution group of Problem I, but also is the best approximation of the matrix group $[\overline{X}_1, \overline{X}_2, \dots, \overline{X}_l]$. About Problem II, we refer the reader to references [5,7–9].

In this paper, absorbing the thought of the conjugate gradient method, and combining the trait of Problem I, we present an iterative method for solving Problem I over bisymmetric matrix group $[X_1, X_2, \dots, X_l]$. By this iterative method, the solvability of Problem I can be judged automatically. When Problem I is consistent, for any initial bisymmetric matrix group

$[X_1^{(0)}, X_2^{(0)}, \ldots, X_l^{(0)}]$, a bisymmetric solution group can be obtained within finite iteration steps in the absence of roundoff errors, and the least norm bisymmetric solution group can be obtained by choosing a special kind of initial bisymmetric matrix group. When Problem I is inconsistent, the inconsistency can also be judged automatically within finite iterative steps. In addition, using this method, the bisymmetric solution group of Problem II can also be obtained by finding the least norm bisymmetric solution group of the new linear matrix equation $A_1\widetilde{X}_1B_1 + A_2\widetilde{X}_2B_2 + \cdots + A_l\widetilde{X}_lB_l = \widetilde{C}$ over real matrices group $[\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_l]$ where $\widetilde{C} = C - A_1\overline{X}_1B_1 - A_1\overline{X}_1B_1 - \cdots - A_l\overline{X}_lB_l$.

2. Iterative methods for solving Problems I and II

In this section, we firstly introduce some lemmas which are required for solving Problem I. We then introduce an iterative method to obtain the bisymmetric solution groups of Problem I. We show that if Problem I is consistent, for any initial bisymmetric matrix group $[X_1^{(0)}, X_2^{(0)}, \ldots, X_l^{(0)}]$, the matrix group sequences $[X_1^{(k)}, X_2^{(k)}, \ldots, X_l^{(k)}]$ generated by the iterative method converge to a bisymmetric solution group within at most pq iteration steps in the absence of roundoff errors, and show that if let above initial bisymmetric matrices be $X_j^{(0)} = A_j^{\mathrm{T}}HB_j^{\mathrm{T}} + B_jHA_j + S_{n_j}(A_j^{\mathrm{T}}HB_j^{\mathrm{T}} + B_jHA_j)S_{n_j}$, $j = 1, 2, \ldots, l$, where H is arbitrary, then the bisymmetric solution group $[X_1^*, X_2^*, \ldots, X_l^*]$ obtained by the iterative method is the least Frobenius norm bisymmetric solution group. Finally, we use the iteration method to solve the Problem II.

Lemma 2.1. *A matrix $X \in BSR^{n\times n}$ if and only if $X = X^{\mathrm{T}} = S_nXS_n$ (see [10]).*

Lemma 2.2. *Suppose that a matrix $X \in SR^{n\times n}$, then $X + S_nXS_n \in BSR^{n\times n}$.*

Proof. Its proof is easy to obtain from Lemma 2.1. □

Lemma 2.3. *Suppose that $A \in R^{n\times n}$, $X \in BSR^{n\times n}$, then,*

$$\langle A, X\rangle = \left\langle \frac{1}{4}[A + A^{\mathrm{T}} + S_n(A + A^{\mathrm{T}})S_n], X\right\rangle.$$

Proof

$$\begin{aligned}
&\left\langle \frac{1}{4}[A + A^{\mathrm{T}} + S_n(A + A^{\mathrm{T}})S_n], X\right\rangle \\
&= \frac{1}{4}[\langle A, X\rangle + \langle A^{\mathrm{T}}, X\rangle + \langle S_nAS_n, X\rangle + \langle S_nA^{\mathrm{T}}S_n, X\rangle] \\
&= \frac{1}{4}[\langle A, X\rangle + \langle A^{\mathrm{T}}, X^{\mathrm{T}}\rangle + \langle A, S_nXS_n\rangle + \langle A^{\mathrm{T}}, S_nXS_n\rangle] \\
&= \frac{1}{4}[\langle A, X\rangle + \langle A^{\mathrm{T}}, X^{\mathrm{T}}\rangle + \langle A, X\rangle + \langle A^{\mathrm{T}}, X^{\mathrm{T}}\rangle] \\
&= \langle A, X\rangle. \qquad \square
\end{aligned}$$

Algorithm 2.1

1. Input matrices $A_i \in R^{p\times n_i}$, $B_i \in R^{n_i\times q}$, $X_i^{(0)} \in BSR^{n_i\times n_i}$ $(i = 1, 2, \ldots, l)$ and $C \in R^{p\times q}$;
2. Calculate
   $R_0 = C - \sum_{i=1}^{l} A_iX_i^{(0)}B_i$;

$Y_{0,i} = A_i^{\mathrm{T}} R_0 B_i^{\mathrm{T}}, i = 1, 2, \dots, l;$
$P_{0,i} = \frac{1}{4}[Y_{0,i} + Y_{0,i}^{\mathrm{T}} + S_{n_i} Y_{0,i} S_{n_i} + S_{n_i} Y_{0,i}^{\mathrm{T}} S_{n_i}], i = 1, 2, \dots, l;$
$k := 0;$

3. If $R_k = 0$, then stop; else, $k := k + 1$;
4. Calculate
$X_i^{(k)} = X_i^{(k-1)} + \frac{\|R_{k-1}\|^2}{\sum_{j=1}^{l} \|P_{k-1,j}\|^2} P_{k-1,i}, i = 1, 2, \dots, l;$
$R_k = C - \sum_{i=1}^{l} A_i X_i^{(k)} B_i = R_{k-1} - \frac{\|R_{k-1}\|^2}{\sum_{j=1}^{l} \|P_{k-1,j}\|^2} \left(\sum_{i=1}^{l} A_i P_{k-1,i} B_i \right);$
$Y_{k,i} = A_i^{\mathrm{T}} R_k B_i^{\mathrm{T}}, i = 1, 2, \dots, l;$
$P_{k,i} = \frac{1}{4}[Y_{k,i} + Y_{k,i}^{\mathrm{T}} + S_{n_i} Y_{k,i} S_{n_i} + S_{n_i} Y_{k,i}^{\mathrm{T}} S_{n_i}] + \frac{\|R_k\|^2}{\|R_{k-1}\|^2} P_{k-1,i}, i = 1, 2, \dots, l;$
5. goto step 3.

Remark 2.1. Obviously, $P_{k,i} \in BSR^{n_i \times n_i}$, $X_i^{(k)} \in BSR^{n_i \times n_i}$ $(i = 1, 2, \dots, l)$ from Algorithm 2.1.

About Algorithm 2.1, we have the following basic properties.

Lemma 2.4. *Suppose that the sequences* $\{R_i\}, \{P_{i,r}\}$ $(R_i \neq 0,\ i = 0, 1, 2, \dots, k, r = 1, 2, \dots, l)$ *are generated by Algorithm* 2.1, *then*

$$\langle R_i, R_j \rangle = 0, \quad \sum_{r=1}^{l} \langle P_{i,r}, P_{j,r} \rangle = 0, \quad (i, j = 0, 1, 2, \dots, k, i \neq j). \tag{2.1}$$

Proof. We prove the conclusion by induction.

Step 1. Show that $\langle R_0, R_1 \rangle = 0$ and $\sum_{r=1}^{l} \langle P_{0,r}, P_{1,r} \rangle = 0$ when $k = 1$,

$$\begin{aligned}
\langle R_0, R_1 \rangle &= \left\langle R_0, R_0 - \frac{\|R_0\|^2}{\sum_{r=1}^{l} \|P_{0,r}\|^2} \sum_{r=1}^{l} A_r P_{0,r} B_r \right\rangle \\
&= \|R_0\|^2 - \frac{\|R_0\|^2}{\sum_{r=1}^{l} \|P_{0,r}\|^2} \sum_{r=1}^{l} \langle A_r^{\mathrm{T}} R_0 B_r^{\mathrm{T}}, P_{0,r} \rangle \\
&= \|R_0\|^2 - \frac{\|R_0\|^2}{\sum_{r=1}^{l} \|P_{0,r}\|^2} \sum_{r=1}^{l} \langle Y_{0,r}, P_{0,r} \rangle \\
&= \|R_0\|^2 - \frac{\|R_0\|^2}{\sum_{r=1}^{l} \|P_{0,r}\|^2} \sum_{r=1}^{l} \left\langle \frac{1}{4}[Y_{0,r} + Y_{0,r}^{\mathrm{T}} + S_{n_r}(Y_{0,r} + Y_{0,r}^{\mathrm{T}}) S_{n_r}], P_{0,r} \right\rangle \\
&= \|R_0\|^2 - \frac{\|R_0\|^2}{\sum_{r=1}^{l} \|P_{0,r}\|^2} \sum_{r=1}^{l} \langle P_{0,r}, P_{0,r} \rangle \\
&= \|R_0\|^2 - \|R_0\|^2 = 0
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{r=1}^{l}\langle P_{0,r}, P_{1,r}\rangle \\
&\quad = \sum_{r=1}^{l}\left\langle P_{0,r}, \frac{1}{4}[Y_{1,r} + Y_{1,r}^{\mathrm{T}} + S_{n_r} Y_{1,r} S_{n_r} + S_{n_r} Y_{1,r}^{\mathrm{T}} S_{n_r}] + \frac{\|R_1\|^2}{\|R_0\|^2} P_{0,r}\right\rangle \\
&\quad = \sum_{r=1}^{l}\left\langle P_{0,r}, \frac{1}{4}[Y_{1,r} + Y_{1,r}^{\mathrm{T}} + S_{n_r} Y_{1,r} S_{n_r} + S_{n_r} Y_{1,r}^{\mathrm{T}} S_{n_r}]\right\rangle + \frac{\|R_1\|^2}{\|R_0\|^2}\sum_{r=1}^{l}\|P_{0,r}\|^2 \\
&\quad = \sum_{r=1}^{l}\langle P_{0,r}, Y_{1,r}\rangle + \frac{\|R_1\|^2}{\|R_0\|^2}\sum_{r=1}^{l}\|P_{0,r}\|^2 \\
&\quad = \sum_{r=1}^{l}\langle P_{0,r}, A_r^{\mathrm{T}} R_1 B_r^{\mathrm{T}}\rangle + \frac{\|R_1\|^2}{\|R_0\|^2}\sum_{r=1}^{l}\|P_{0,r}\|^2 \\
&\quad = \left\langle \sum_{r=1}^{l} A_r P_{0,r} B_r, R_1\right\rangle + \frac{\|R_1\|^2}{\|R_0\|^2}\sum_{r=1}^{l}\|P_{0,r}\|^2 \\
&\quad = \frac{\sum_{r=1}^{l}\|P_{0,r}\|^2}{\|R_0\|^2}\langle R_0 - R_1, R_1\rangle + \frac{\|R_1\|^2}{\|R_0\|^2}\sum_{r=1}^{l}\|P_{0,r}\|^2 \\
&\quad = -\frac{\sum_{r=1}^{l}\|P_{0,r}\|^2}{\|R_0\|^2}\|R_1\|^2 + \frac{\|R_1\|^2}{\|R_0\|^2}\sum_{r=1}^{l}\|P_{0,r}\|^2 \\
&\quad = 0.
\end{aligned}
$$

Step 2. Suppose that (2.1) holds when $k = s$, then when $k = s + 1$,

$$
\begin{aligned}
&\langle R_s, R_{s+1}\rangle \\
&\quad = \left\langle R_s, R_s - \frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l} A_r P_{s,r} B_r\right\rangle \\
&\quad = \|R_s\|^2 - \frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l}\langle A_r^{\mathrm{T}} R_s B_r^{\mathrm{T}}, P_{s,r}\rangle \\
&\quad = \|R_s\|^2 - \frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l}\langle Y_{s,r}, P_{s,r}\rangle \\
&\quad = \|R_s\|^2 - \frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l}\left\langle \frac{1}{4}[Y_{s,r} + Y_{s,r}^{\mathrm{T}} + S_{n_r}(Y_{s,r} + Y_{s,r}^{\mathrm{T}}) S_{n_r}], P_{s,r}\right\rangle \\
&\quad = \|R_s\|^2 - \frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l}\left\langle P_{s,r} - \frac{\|R_s\|^2}{\|R_{s-1}\|^2} P_{s-1,r}, P_{s,r}\right\rangle \\
&\quad = \|R_s\|^2 - \|R_s\|^2 = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{r=1}^{l}\langle P_{s,r}, P_{s+1,r}\rangle \\
&\quad= \sum_{r=1}^{l}\left\langle P_{s,r}, \frac{1}{4}[Y_{s+1,r} + Y_{s+1,r}^{\mathrm{T}} + S_{n_r}Y_{s+1,r}S_{n_r} + S_{n_r}Y_{s+1,r}^{\mathrm{T}}S_{n_r}] + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}P_{s,r}\right\rangle \\
&\quad= \sum_{r=1}^{l}\left\langle P_{s,r}, \frac{1}{4}[Y_{s+1,r} + Y_{s+1,r}^{\mathrm{T}} + S_{n_r}Y_{s+1,r}S_{n_r} + S_{n_r}Y_{s+1,r}^{\mathrm{T}}S_{n_r}]\right\rangle \\
&\qquad+\frac{\|R_{s+1}\|^2}{\|R_s\|^2}\sum_{r=1}^{l}\langle P_{s,r}, P_{s,r}\rangle \\
&\quad= \sum_{r=1}^{l}\langle P_{s,r}, Y_{s+1,r}\rangle + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\sum_{r=1}^{l}\langle P_{s,r}, P_{s,r}\rangle \\
&\quad= \sum_{r=1}^{l}\langle P_{s,r}, A_r^{\mathrm{T}}R_{s+1}B_r^{\mathrm{T}}\rangle + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\sum_{r=1}^{l}\langle P_{s,r}, P_{s,r}\rangle \\
&\quad= \left\langle \sum_{r=1}^{l}A_rP_{s,r}B_r, R_{s+1}\right\rangle + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\sum_{r=1}^{l}\|P_{s,r}\|^2 \\
&\quad= \frac{\sum_{r=1}^{l}\|P_{s,r}\|^2}{\|R_s\|^2}\langle R_s - R_{s+1}, R_{s+1}\rangle + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\sum_{r=1}^{l}\|P_{s,r}\|^2 \\
&\quad= -\frac{\sum_{r=1}^{l}\|P_{s,r}\|^2}{\|R_s\|^2}\|R_{s+1}\|^2 + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\sum_{r=1}^{l}\|P_{s,r}\|^2 \\
&\quad= 0.
\end{aligned}
$$

For $j = 1, 2, \ldots, s-1$, we have that

$$
\begin{aligned}
&\langle R_j, R_{s+1}\rangle \\
&\quad= \left\langle R_j, R_s - \frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l}A_rP_{s,r}B_r\right\rangle \\
&\quad= -\frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l}\langle A_r^{\mathrm{T}}R_jB_r^{\mathrm{T}}, P_{s,r}\rangle \\
&\quad= -\frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l}\langle Y_{j,r}, P_{s,r}\rangle \\
&\quad= -\frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l}\left\langle \frac{1}{4}[Y_{j,r} + Y_{j,r}^{\mathrm{T}} + S_{n_r}(Y_{j,r} + Y_{j,r}^{\mathrm{T}})S_{n_r}], P_{s,r}\right\rangle \\
&\quad= -\frac{\|R_s\|^2}{\sum_{r=1}^{l}\|P_{s,r}\|^2}\sum_{r=1}^{l}\left\langle P_{j,r} - \frac{\|R_j\|^2}{\|R_{j-1}\|^2}P_{j-1,r}, P_{s,r}\right\rangle \\
&\quad= 0
\end{aligned}
$$

and

$$\begin{aligned}
&\sum_{r=1}^{l}\langle P_{j,r}, P_{s+1,r}\rangle\\
&\quad= \sum_{r=1}^{l}\left\langle P_{j,r}, \frac{1}{4}[Y_{s+1,r}+Y_{s+1,r}^{\mathrm{T}}+S_{n_r}Y_{s+1,r}S_{n_r}+S_{n_r}Y_{s+1,r}^{\mathrm{T}}S_{n_r}]+\frac{\|R_{s+1}\|^2}{\|R_s\|^2}P_{s,r}\right\rangle\\
&\quad= \sum_{r=1}^{l}\left\langle P_{j,r}, \frac{1}{4}[Y_{s+1,r}+Y_{s+1,r}^{\mathrm{T}}+S_{n_r}Y_{s+1,r}S_{n_r}+S_{n_r}Y_{s+1,r}^{\mathrm{T}}S_{n_r}]\right\rangle\\
&\qquad+\frac{\|R_{s+1}\|^2}{\|R_s\|^2}\sum_{r=1}^{l}\langle P_{j,r}, P_{s,r}\rangle\\
&\quad= \sum_{r=1}^{l}\langle P_{j,r}, Y_{s+1,r}\rangle\\
&\quad= \sum_{r=1}^{l}\langle P_{j,r}, A_r^{\mathrm{T}}R_{s+1}B_r^{\mathrm{T}}\rangle\\
&\quad= \left\langle \sum_{r=1}^{l}A_rP_{j,r}B_r, R_{s+1}\right\rangle\\
&\quad= \frac{\sum_{r=1}^{l}\|P_{j,r}\|^2}{\|R_j\|^2}\langle R_j-R_{j+1}, R_{s+1}\rangle\\
&\quad= 0.
\end{aligned}$$

From steps 1 and 2, the conclusion $\langle R_i, R_j\rangle = 0$ and $\sum_{r=1}^{l}\langle P_{i,r}, P_{j,r}\rangle = 0$ hold for all $i, j = 0, 1, 2, \ldots, k (i \neq j)$ by the principle of induction. □

Lemma 2.5. *Suppose that Problem* I *is consistent, and* $[X_1^*, X_2^*, \ldots, X_l^*]$ *is a bisymmetric solution group, then, for any initial bisymmetric matrix group* $[X_1^{(0)}, X_2^{(0)}, \ldots, X_l^{(0)}]$, *the sequences* $\{X_j^{(i)}\}, \{R_i\}, \{P_{i,j}\}$ $(j = 1, 2, \ldots, l)$ *generated by Algorithm* 2.1 *satisfy*

$$\sum_{j=1}^{l}\langle P_{i,j}, X_j^*-X_j^{(i)}\rangle = \|R_i\|^2 \quad (i = 0, 1, 2, \ldots).$$

Proof. We prove the conclusion by induction. When $i = 0$,

$$\begin{aligned}
&\sum_{j=1}^{l}\langle P_{0,j}, X_j^*-X_j^{(0)}\rangle\\
&\quad= \sum_{j=1}^{l}\left\langle \frac{1}{4}[Y_{0,j}+Y_{0,j}^{\mathrm{T}}+S_{n_j}Y_{0,j}S_{n_j}+S_{n_j}Y_{0,j}^{\mathrm{T}}S_{n_j}], X_j^*-X_j^{(0)}\right\rangle\\
&\quad= \sum_{j=1}^{l}\langle Y_{0,j}, X_j^*-X_j^{(0)}\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{l}\langle A_j^{\mathrm{T}} R_0 B_j^{\mathrm{T}}, X_j^* - X_j^{(0)}\rangle \\
&= \sum_{j=1}^{l}\langle R_0, A_j(X_j^* - X_j^{(0)})B_j\rangle \\
&= \left\langle R_0, \sum_{j=1}^{l} A_j X_j^* B_j - \sum_{j=1}^{l} A_j X_j^{(0)} B_j \right\rangle \\
&= \langle R_0, R_0\rangle \\
&= \|R_0\|^2.
\end{aligned}
$$

Suppose that the conclusion holds for $i = s(s \geqslant 0)$, that is, $\sum_{j=1}^{l}\langle P_{s,j}, X_j^* - X_j^{(s)}\rangle = \|R_s\|^2$, then, when $i = s+1$,

$$
\begin{aligned}
&\sum_{j=1}^{l}\langle P_{s+1,j}, X_j^* - X_j^{(s+1)}\rangle \\
&\quad= \sum_{j=1}^{l}\left\langle \frac{1}{4}[Y_{s+1,j} + Y_{s+1,j}^{\mathrm{T}} + S_{n_j} Y_{s+1,j} S_{n_j} + S_{n_j} Y_{s+1,j}^{\mathrm{T}} S_{n_j}] \right. \\
&\qquad \left. + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} P_{s,j}, X_j^* - X_j^{(s+1)} \right\rangle \\
&\quad= \sum_{j=1}^{l}\left\langle \frac{1}{4}[Y_{s+1,j} + Y_{s+1,j}^{\mathrm{T}} + S_{n_j} Y_{s+1,j} S_{n_j} + S_{n_j} Y_{s+1,j}^{\mathrm{T}} S_{n_j}], X_j^* - X_j^{(s+1)} \right\rangle \\
&\qquad + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \sum_{j=1}^{l}\langle P_{s,j}, X_j^* - X_j^{(s+1)}\rangle \\
&\quad= \sum_{j=1}^{l}\langle Y_{s+1,j}, X_j^* - X_j^{(s+1)}\rangle + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \sum_{j=1}^{l}\langle P_{s,j}, X_j^* - X_j^{(s+1)}\rangle \\
&\quad= \sum_{j=1}^{l}\langle A_j^{\mathrm{T}} R_{s+1} B_j^{\mathrm{T}}, X_j^* - X_j^{(s+1)}\rangle + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \sum_{j=1}^{l}\langle P_{s,j}, X_j^* - X_j^{(s+1)}\rangle \\
&\quad= \sum_{j=1}^{l}\langle R_{s+1}, A_j(X_j^* - X_j^{(s+1)})B_j\rangle + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \sum_{j=1}^{l}\langle P_{s,j}, X_j^* - X_j^{(s+1)}\rangle \\
&\quad= \|R_{s+1}\|^2 + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \sum_{j=1}^{l}\left\langle P_{s,j}, X_j^* - X_j^{(s)} - \frac{\|R_s\|^2}{\sum_{j=1}^{l}\|P_{s,j}\|^2} P_{s,j} \right\rangle \\
&\quad= \|R_{s+1}\|^2 + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \left[\sum_{j=1}^{l}\langle P_{s,j}, X_j^* - X_j^{(s)}\rangle - \|R_s\|^2 \right] = \|R_{s+1}\|^2.
\end{aligned}
$$

By the principle of induction, the conclusion $\sum_{j=1}^{l}\langle P_{i,j}, X_j^* - X_j^{(i)}\rangle = \|R_i\|^2$ holds for all $i = 0, 1, 2, \ldots$. $\square$

Theorem 2.1. *Suppose that Problem* I *is consistent, then for any initial bisymmetric matrix group* $[X_1^{(0)}, X_2^{(0)}, \ldots, X_l^{(0)}]$, *a bisymmetric solution group can be obtained within at most* pq *iteration steps by Algorithm* 2.1.

Proof. If $R_i \neq 0\,(i = 0, 1, 2, \ldots, pq-1)$, then $P_{i,j} \neq 0$ for some $j \in \{1, 2, \ldots, l\}$ from Lemma 2.5. Hence, $[X_1^{pq}, X_2^{pq}, \ldots, X_l^{pq}]$ and R_{pq} can be obtained by Algorithm 2.1. From Lemma 2.4, we have that

$$\langle R_i, R_{pq}\rangle = 0, \quad i = 0, 1, 2, \ldots, pq-1$$

and

$$\langle R_i, R_j\rangle = 0, \quad i, j = 0, 1, 2, \ldots, pq-1, i \neq j.$$

So the set of $R_0, R_1, \ldots, R_{pq-1}$ is an orthogonal basis of the matrix space $R^{p\times q}$, which implies that $R_{pq} = 0$, i.e. $[X_1^{(pq)}, X_2^{(pq)}, \ldots, X_l^{(pq)}]$ is a bisymmetric solution group of Problem I. $\square$

Theorem 2.2. *Problem* I *is consistent if and only if there exists a nonnegative integer number* k, *such that* $R_k = 0$ *or* $P_{k,j} \neq 0$ *for some* $j \in \{1, 2, \ldots, l\}$.

Proof. Suppose that there exists a nonnegative integer number k, such that $R_k = 0$, then Problem I is obviously consistent. If $P_{k,j} \neq 0$ for some $j \in \{1, 2, \ldots, l\}$, then a bisymmetric solution group of Problem I can be obtained within at most pq iteration steps from the proof process of Theorem 2.1, so Problem I is also consistent.

Conversely, suppose that Problem I is consistent, then there exists a nonnegative integer number k, such that $R_k = 0$ or $P_{k,j} \neq 0$ for some $j \in \{1, 2, \ldots, l\}$. Actually, if $R_k \neq 0$, $P_{k,j} = 0$ for all $j \in \{1, 2, \ldots, l\}$, then it contradicts to Lemma 2.5. $\square$

Remark 2.2. From Lemma 2.5, if there exists a nonnegative integer number k such that $P_{k,j} = 0$ for all $j \in \{1, 2, \ldots, l\}$, but $R_k \neq 0$, then Problem I is inconsistent. Hence, the solvability of Problem I can be judged automatically by Algorithm 2.1.

Lemma 2.6. *Problem* I *has bisymmetric solution groups if and only if the following linear matrix equations is consistent*:

$$\begin{cases} A_1X_1B_1 + A_2X_2B_2 + \cdots + A_lX_lB_l = C, \\ B_1^{\mathrm{T}}X_1A_1^{\mathrm{T}} + B_2^{\mathrm{T}}X_2A_2^{\mathrm{T}} + \cdots + B_l^{\mathrm{T}}X_lA_l^{\mathrm{T}} = C^{\mathrm{T}}, \\ A_1S_{n_1}X_1S_{n_1}B_1 + A_2S_{n_2}X_2S_{n_2}B_2 + \cdots + A_lS_{n_l}X_lS_{n_l}B_l = C, \\ B_1^{\mathrm{T}}S_{n_1}X_1S_{n_1}A_1^{\mathrm{T}} + B_2^{\mathrm{T}}S_{n_2}X_2S_{n_2}A_2^{\mathrm{T}} + \cdots + B_l^{\mathrm{T}}S_{n_l}X_lS_{n_l}A_l^{\mathrm{T}} = C^{\mathrm{T}}. \end{cases} \tag{2.2}$$

Proof. Suppose that Problem I has a bisymmetric solution group $[Y_1, Y_2, \ldots, Y_l]$, then $Y_i = Y_i^{\mathrm{T}} = S_{n_i}Y_iS_{n_i}$ $(i = 1, 2, \ldots, l)$, and

$$\begin{aligned} &A_1Y_1B_1 + A_2Y_2B_2 + \cdots + A_lY_lB_l = C, \\ &B_1^{\mathrm{T}}Y_1A_1^{\mathrm{T}} + B_2^{\mathrm{T}}Y_2A_2^{\mathrm{T}} + \cdots + B_l^{\mathrm{T}}Y_lA_l^{\mathrm{T}} \\ &\quad = (A_1Y_1^{\mathrm{T}}B_1 + A_2Y_2^{\mathrm{T}}B_2 + \cdots + A_lY_l^{\mathrm{T}}B_l)^{\mathrm{T}} \end{aligned}$$

$$
\begin{aligned}
&= (A_1Y_1B_1 + A_2Y_2B_2 + \cdots + A_lY_lB_l)^{\mathrm{T}}\\
&= C^{\mathrm{T}},\\
&A_1S_{n_1}Y_1S_{n_1}B_1 + A_2S_{n_2}Y_2S_{n_2}B_2 + \cdots + A_lS_{n_l}Y_lS_{n_l}B_l\\
&= A_1Y_1B_1 + A_2Y_2B_2 + \cdots + A_lY_lB_l\\
&= C,\\
&B_1^{\mathrm{T}}S_{n_1}Y_1S_{n_1}A_1^{\mathrm{T}} + B_2^{\mathrm{T}}S_{n_2}Y_2S_{n_2}A_2^{\mathrm{T}} + \cdots + B_l^{\mathrm{T}}S_{n_l}Y_lS_{n_l}A_l^{\mathrm{T}}\\
&= B_1^{\mathrm{T}}Y_1^{\mathrm{T}}A_1^{\mathrm{T}} + B_2^{\mathrm{T}}Y_2^{\mathrm{T}}A_2^{\mathrm{T}} + \cdots + B_l^{\mathrm{T}}Y_l^{\mathrm{T}}A_l^{\mathrm{T}}\\
&= (A_1Y_1B_1 + A_2Y_2B_2 + \cdots + A_lY_lB_l)^{\mathrm{T}}\\
&= C^{\mathrm{T}}.
\end{aligned}
$$

Hence, the bisymmetric solution group $[Y_1, Y_2, \ldots, Y_l]$ is a solution group of the linear matrix equations (2.2), that is, the linear matrix equations (2.2) is consistent.

Conversely, suppose that the linear matrix equations (2.2) is consistent, then there exists a matrix group $[\overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_l](\overline{Y}_i \in R^{n_i\times n_i}, i = 1, 2, \ldots, l)$, such that

$$
\begin{cases}
A_1\overline{Y}_1B_1 + A_2\overline{Y}_2B_2 + \cdots + A_l\overline{Y}_lB_l = C,\\
B_1^{\mathrm{T}}\overline{Y}_1A_1^{\mathrm{T}} + B_2^{\mathrm{T}}\overline{Y}_2A_2^{\mathrm{T}} + \cdots + B_l^{\mathrm{T}}\overline{Y}_lA_l^{\mathrm{T}} = C^{\mathrm{T}},\\
A_1S_{n_1}\overline{Y}_1S_{n_1}B_1 + A_2S_{n_2}\overline{Y}_2S_{n_2}B_2 + \cdots + A_lS_{n_l}\overline{Y}_lS_{n_l}B_l = C,\\
B_1^{\mathrm{T}}S_{n_1}\overline{Y}_1S_{n_1}A_1^{\mathrm{T}} + B_2^{\mathrm{T}}S_{n_2}\overline{Y}_2S_{n_2}A_2^{\mathrm{T}} + \cdots + B_l^{\mathrm{T}}S_{n_l}\overline{Y}_lS_{n_l}A_l^{\mathrm{T}} = C^{\mathrm{T}}.
\end{cases}
$$

Let $Y_i = \frac{\overline{Y}_i+\overline{Y}_i^{\mathrm{T}}+S_{n_i}(\overline{Y}_i+\overline{Y}_i^{\mathrm{T}})S_{n_i}}{4}$, then $Y_i \in BSR^{n\times n}$, and

$$
\begin{aligned}
&A_1Y_1B_1 + A_2Y_2B_2 + \cdots + A_lY_lB_l\\
&= A_1\frac{\overline{Y}_1+\overline{Y}_1^{\mathrm{T}}+S_{n_1}(\overline{Y}_1+\overline{Y}_1^{\mathrm{T}})S_{n_1}}{4}B_1 + A_2\frac{\overline{Y}_2+\overline{Y}_2^{\mathrm{T}}+S_{n_2}(\overline{Y}_2+\overline{Y}_2^{\mathrm{T}})S_{n_2}}{4}B_2\\
&\quad + \cdots + A_l\frac{\overline{Y}_l+\overline{Y}_l^{\mathrm{T}}+S_{n_l}(\overline{Y}_l+\overline{Y}_l^{\mathrm{T}})S_{n_l}}{4}B_l\\
&= \frac{\sum_{i=1}^{l} A_i\overline{Y}_iB_i + \left(\sum_{i=1}^{l} B_i^{\mathrm{T}}\overline{Y}_iA_i^{\mathrm{T}}\right)^{\mathrm{T}} + \sum_{i=1}^{l} A_iS_{n_i}\overline{Y}_iS_{n_i}B_i + \left(\sum_{i=1}^{l} B_i^{\mathrm{T}}S_{n_i}\overline{Y}_iS_{n_i}A_i^{\mathrm{T}}\right)^{\mathrm{T}}}{4}\\
&= \frac{C+C+C+C}{4}\\
&= C.
\end{aligned}
$$

Therefore, $[Y_1, Y_2, \ldots, Y_l]$ is a bisymmetric solution group of Problem I. $\square$

Remark 2.3. From the proof process of Lemma 2.6, any bisymmetric solution groups of Problem I must be the solution groups of the linear matrix equations (2.2). If let S_E' denote the solution group set of the linear matrix equations (2.2), then $S_E \subseteq S_{E'}$, where S_E is the solution group set of Problem I. Therefore, if we want to prove that $[X_1^*, X_2^*, \ldots, X_l^*]$ is the least Frobenius norm bisymmetric solution group of Problem I, then, it is enough to prove that $[X_1^*, X_2^*, \ldots, X_l^*]$ is the least Frobenius norm bisymmetric solution group of the linear matrix equations (2.2).

Lemma 2.7. *Suppose that the consistent systems of the linear equations* $Ax = b$ *has a solution* $x^* \in R(A^{\mathrm{T}})$, *then* x^* *is an unique least Frobenius norm solution of the systems of linear equations.*

Proof. We decompose the matrix $A \in R^{m\times n}$ by SVD:

$$A = U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^{\mathrm{T}} = U_1\Sigma V_1^{\mathrm{T}},$$

where $U = (U_1, U_2) \in OR^{m\times m}, V = (V_1, V_2) \in OR^{n\times n}, U_1 \in R^{m\times r}, V_1 \in R^{n\times r}, \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) > 0, r = \mathrm{rank}(A)$, then the Moore–Penrose generalized inverse of matrix A is that $A^+ = V_1\Sigma^{-1}U_1^{\mathrm{T}}$, and the general solution of the system of linear equation $Ax = b$ is that $x = A^+b + (I - A^+A)z$, where z is an arbitrary vector with suitable dimension. Since $A^+b = V_1\Sigma^{-1}U_1^{\mathrm{T}}b \in R(V_1)$, $(I - A^+A)z = V_2V_2^{\mathrm{T}}z \in R(V_2)$, V_1 and V_2 are orthogonal to each other, then A^+b is the unique least Frobenius norm solution of the system of linear equations $Ax = b$. On the other hand, since $A^{\mathrm{T}} = V_1\Sigma U_1^{\mathrm{T}}$, and the solution $x^* \in R(A^{\mathrm{T}})$, then $x^* \in R(V_1)$. Therefore, x^* is the least Frobenius norm solution of the system of linear equations $Ax = b$, that is, $x^* = A^+b$. $\square$

Theorem 2.3. *Suppose that Problem* I *is consistent. If we choose the initial bisymmetric matrices* $X_j^{(0)} = A_j^{\mathrm{T}}HB_j^{\mathrm{T}} + B_jH^{\mathrm{T}}A_j + S_{n_j}(A_j^{\mathrm{T}}HB_j^{\mathrm{T}} + B_jH^{\mathrm{T}}A_j)S_{n_j},\ \ j = 1, 2, \ldots, l,\ H$ *is arbitrary, or more especially, let* $X_1^{(0)} = 0, X_2^{(0)} = 0, \ldots, X_l^{(0)} = 0$, *then the bisymmetric solution group* $[X_1^*, X_2^*, \ldots, X_l^*]$ *obtained by Algorithm* 2.1 *is the unique least Frobenius norm bisymmetric solution group of Problem* I.

Proof. From Theorem 2.1, if we take $X_i^{(0)} = A_i^{\mathrm{T}}HB_i^{\mathrm{T}} + B_iH^{\mathrm{T}}A_i + S_{n_i}(A_i^{\mathrm{T}}HB_i^{\mathrm{T}} + B_iH^{\mathrm{T}}A_i)S_{n_i}$ $(i = 1, 2, \ldots, l)$ (H is arbitrary), we can obtain the bisymmetric solution group $[X_1^*, X_2^*, \ldots, X_l^*]$ of Problem I within finite iteration steps, and $X_i^*(i = 1, 2, \ldots, l)$ can be expressed as

$$X_i^* = A_i^{\mathrm{T}}YB_i^{\mathrm{T}} + B_iY^{\mathrm{T}}A_i + S_{n_i}(A_i^{\mathrm{T}}YB_i^{\mathrm{T}} + B_iY^{\mathrm{T}}A_i)S_{n_i},$$

where $Y \in R^{p\times q}$. In the sequel, we will prove that $[X_1^*, X_2^*, \ldots, X_l^*]$ is the unique least Frobenius norm bisymmetric solution group of Problem I. From the Remark 2.3, we only prove that $[X_1^*, X_2^*, \ldots, X_l^*]$ is the least Frobenius norm bisymmetric solution group of the linear matrix equations (2.2).

For matrix $A \in R^{m\times n}$, let $\mathrm{vec}(A)$ denote the following mn-vector containing all the entries of matrix A:

$$\mathrm{vec}(A) = \left(A(:,1)^{\mathrm{T}}A(:,2)^{\mathrm{T}}\cdots A(:,n)^{\mathrm{T}}\right)^{\mathrm{T}} \in R^{mn},$$

where $A(:,i)$ denote ith column of matrix A (i.e., Matlab style). $A \otimes B$ denote the Kronecker product of matrices A and B. Then linear matrix equations (2.2) is equivalent to the system of linear matrix equations

$$\begin{pmatrix} B_1^{\mathrm{T}} \otimes A_1 & B_2^{\mathrm{T}} \otimes A_2 & \cdots & B_l^{\mathrm{T}} \otimes A_l\\ A_1 \otimes B_1^{\mathrm{T}} & A_2 \otimes B_2^{\mathrm{T}} & \cdots & A_l \otimes B_l^{\mathrm{T}}\\ (B_1^{\mathrm{T}}S_{n_1}) \otimes (A_1S_{n_1}) & (B_2^{\mathrm{T}}S_{n_2}) \otimes (A_2S_{n_2}) & \cdots & (B_l^{\mathrm{T}}S_{n_l}) \otimes (A_lS_{n_l})\\ (A_1S_{n_1}) \otimes (B_1^{\mathrm{T}}S_{n_1}) & (A_2S_{n_2}) \otimes (B_2^{\mathrm{T}}S_{n_2}) & \cdots & (A_lS_{n_l}) \otimes (B_l^{\mathrm{T}}S_{n_l})\end{pmatrix}$$

$$\times \begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \vdots \\ \mathrm{vec}(X_l) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}(C) \\ \mathrm{vec}(C^{\mathrm{T}}) \\ \mathrm{vec}(C) \\ \mathrm{vec}(C^{\mathrm{T}}) \end{pmatrix}. \tag{2.3}$$

Noting that

$$\begin{aligned}
\begin{pmatrix} \mathrm{vec}(X_1^*) \\ \mathrm{vec}(X_2^*) \\ \vdots \\ \mathrm{vec}(X_l^*) \end{pmatrix} &= \begin{pmatrix} \mathrm{vec}(A_1^{\mathrm{T}} Y B_1^{\mathrm{T}} + B_1 Y^{\mathrm{T}} A_1 + S_{n_1}(A_1^{\mathrm{T}} Y B_1^{\mathrm{T}} + B_1 Y^{\mathrm{T}} A_1) S_{n_1}) \\ \mathrm{vec}(A_2^{\mathrm{T}} Y B_2^{\mathrm{T}} + B_2 Y^{\mathrm{T}} A_2 + S_{n_2}(A_2^{\mathrm{T}} Y B_2^{\mathrm{T}} + B_2 Y^{\mathrm{T}} A_2) S_{n_2}) \\ \vdots \\ \mathrm{vec}(A_l^{\mathrm{T}} Y B_l^{\mathrm{T}} + B_l Y^{\mathrm{T}} A_l + S_{n_l}(A_l^{\mathrm{T}} Y B_l^{\mathrm{T}} + B_l Y^{\mathrm{T}} A_l) S_{n_l}) \end{pmatrix} \\
&= \begin{pmatrix} B_1 \otimes A_1^{\mathrm{T}} & A_1^{\mathrm{T}} \otimes B_1 & (S_{n_1} B_1) \otimes (S_{n_1} A_1^{\mathrm{T}}) & (S_{n_1} A_1^{\mathrm{T}}) \otimes (S_{n_1} B_1) \\ B_2 \otimes A_2^{\mathrm{T}} & A_2^{\mathrm{T}} \otimes B_2 & (S_{n_2} B_2) \otimes (S_{n_2} A_2^{\mathrm{T}}) & (S_{n_2} A_2^{\mathrm{T}}) \otimes (S_{n_2} B_2) \\ \vdots & \vdots & \vdots & \vdots \\ B_l \otimes A_l^{\mathrm{T}} & A_l^{\mathrm{T}} \otimes B_l & (S_{n_l} B_l) \otimes (S_{n_l} A_l^{\mathrm{T}}) & (S_{n_l} A_l^{\mathrm{T}}) \otimes (S_{n_l} B_l) \end{pmatrix} \begin{pmatrix} \mathrm{vec}(Y) \\ \mathrm{vec}(Y^{\mathrm{T}}) \\ \mathrm{vec}(Y) \\ \mathrm{vec}(Y^{\mathrm{T}}) \end{pmatrix} \\
&= \begin{pmatrix} B_1^{\mathrm{T}} \otimes A_1 & B_2^{\mathrm{T}} \otimes A_2 & \cdots & B_l^{\mathrm{T}} \otimes A_l \\ A_1 \otimes B_1^{\mathrm{T}} & A_2 \otimes B_2^{\mathrm{T}} & \cdots & A_l \otimes B_l^{\mathrm{T}} \\ (B_1^{\mathrm{T}} S_{n_1}) \otimes (A_1 S_{n_1}) & (B_2^{\mathrm{T}} S_{n_2}) \otimes (A_2 S_{n_2}) & \cdots & (B_l^{\mathrm{T}} S_{n_l}) \otimes (A_{n_l} S_l) \\ (A_1 S_{n_1}) \otimes (B_1^{\mathrm{T}} S_{n_1}) & (A_2 S_{n_2}) \otimes (B_2^{\mathrm{T}} S_{n_2}) & \cdots & (A_l S_{n_l}) \otimes (B_l^{\mathrm{T}} S_{n_l}) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathrm{vec}(Y) \\ \mathrm{vec}(Y^{\mathrm{T}}) \\ \mathrm{vec}(Y) \\ \mathrm{vec}(Y^{\mathrm{T}}) \end{pmatrix} \\
&\in R\left(\begin{pmatrix} B_1^{\mathrm{T}} \otimes A_1 & B_2^{\mathrm{T}} \otimes A_2 & \cdots & B_l^{\mathrm{T}} \otimes A_l \\ A_1 \otimes B_1^{\mathrm{T}} & A_2 \otimes B_2^{\mathrm{T}} & \cdots & A_l \otimes B_l^{\mathrm{T}} \\ (B_1^{\mathrm{T}} S_{n_1}) \otimes (A_1 S_{n_1}) & (B_2^{\mathrm{T}} S_{n_2}) \otimes (A_2 S_{n_2}) & \cdots & (B_l^{\mathrm{T}} S_{n_l}) \otimes (A_l S_{n_l}) \\ (A_1 S_{n_1}) \otimes (B_1^{\mathrm{T}} S_{n_1}) & (A_2 S_{n_2}) \otimes (B_2^{\mathrm{T}} S_{n_2}) & \cdots & (A_l S_{n_l}) \otimes (B_l^{\mathrm{T}} S_{n_l}) \end{pmatrix}^{\mathrm{T}} \right).
\end{aligned}$$

Hence, from Lemma 2.7, $[\mathrm{vec}(X_1^*), \mathrm{vec}(X_2^*), \ldots, \mathrm{vec}(X_l^*)]$ is the unique least Frobenius norm bisymmetric solution group of the matrix equations (2.3). Since vec operator is isomorphic, $[X_1^*, X_2^*, \ldots, X_l^*]$ is the unique least Frobenius norm bisymmetric solution group of the linear matrix equations (2.2), thus, it is also the unique least Frobenius norm bisymmetric solution group of Problem I. □

When Problem I is consistent, its bisymmetric solution group set S_E is nonempty, then

$$\begin{aligned}
&A_1 X_1 B_1 + A_2 X_2 B_2 + \cdots + A_l X_l B_l = C \\
&\quad \Leftrightarrow A_1(X_1 - \overline{X}_1) B_1 + A_2(X_2 - \overline{X}_2) B_2 + \cdots + A_l(X_l - \overline{X}_l) B_l \\
&\quad = C - A_1 \overline{X}_1 B_1 - A_2 \overline{X}_2 B_2 - \cdots - A_l \overline{X}_l B_l.
\end{aligned}$$

Let $\widetilde{X}_1 = X_1 - \overline{X}_1, \widetilde{X}_2 = X_2 - \overline{X}_2, \ldots, \widetilde{X}_l = X_l - \overline{X}_l$ and $\widetilde{C} = C - A_1 \overline{X}_1 B_1 - A_2 \overline{X}_2 B_2 - \cdots - A_l \overline{X}_l B_l$, then Problem II is equivalent to find the least Frobenius norm bisymmetric solution group of the linear matrix equation

$$A_1 \widetilde{X}_1 B_1 + A_2 \widetilde{X}_2 B_2 + \cdots + A_l \widetilde{X}_l B_l = \widetilde{C}. \tag{2.4}$$

By using the Algorithm 2.1, let initial matrices $\widetilde{X}_j^{(0)} = A_j^{\mathrm{T}} H B_j^{\mathrm{T}} + B_j H^{\mathrm{T}} A_j + S_{n_j}(A_j^{\mathrm{T}} H B_j^{\mathrm{T}} + B_j H^{\mathrm{T}} A_j) S_{n_j}$, $j = 1, 2, \ldots, l$, where H is arbitrary, or more especially, let $\widetilde{X}_1^{(0)} = 0$, $\widetilde{X}_2^{(0)} = 0, \ldots, \widetilde{X}_l^{(0)} = 0$, we can obtain the unique least Frobenius norm bisymmetric solution group $[\widetilde{X}_1^*, \widetilde{X}_2^*, \ldots, \widetilde{X}_l^*]$ of the linear matrix equation (2.4). Once above bisymmetric matrix group

$[\widetilde{X}_1^*, \widetilde{X}_2^*, \ldots, \widetilde{X}_l^*]$ is obtained, the unique bisymmetric solution group $[\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_l]$ of problem II can be obtained. In this case, $\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_l$ can be expressed as $\widehat{X}_1 = \widetilde{X}_1^* + \overline{X}_1$, $\widehat{X}_2 = \widetilde{X}_2^* + \overline{X}_2, \ldots, \widehat{X}_l = \widetilde{X}_l^* + \overline{X}_l$.

3. Conclusions

In this paper, we first introduce an iterative method, that is, Algorithm 2.1 for solving Problem I. We then show that if Problem I is consistent, for any initial bisymmetric matrix group $[X_1^{(0)}, X_2^{(0)}, \ldots, X_l^{(0)}]$, the bisymmetric matrix group sequence $\{[X_1^{(k)}, X_2^{(k)}, \ldots, X_l^{(k)}]\}$ generated by Algorithm 2.1 converges to its a bisymmetric solution group within at most pq iteration steps in the absence of roundoff errors, and show that if let the above initial matrices be $X_j^{(0)} = A_j^{\mathrm{T}} H B_j^{\mathrm{T}} + B_j H A_j + S_{n_j}(A_j^{\mathrm{T}} H B_j^{\mathrm{T}} + B_j H A_j) S_{n_j}$, $j = 1, 2, \ldots, l$, where H is arbitrary, then the bisymmetric solution group $[X_1^*, X_2^*, \ldots, X_l^*]$ obtained by the iterative method is the least Frobenius norm bisymmetric solution group. Finally, we consider using Algorithm 2.1 to solve Problem II.

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